TEKS G.6E

Examining Proofs

(6) Proof and congruence. The student uses the process skills with deductive reasoning to prove and apply theorems by using a variety of methods such as coordinate, transformational, and axiomatic and formats such as two-column, paragraph, and flow chart. The student is expected to:

(A) verify theorems about angles formed by the intersection of lines and line segments, including vertical angles, and angles formed by parallel lines cut by a transversal and prove equidistance between the endpoints of a segment and points on its perpendicular bisector and apply these relationships to solve problems;

(B) prove two triangles are congruent by applying the Side-Angle-Side, Angle-Side-Angle, Side-Side-Side, Angle-Angle-Side, and Hypotenuse-Leg congruence conditions;

(C) apply the definition of congruence, in terms of rigid transformations, to identify congruent figures and their corresponding sides and angles;

(D) verify theorems about the relationships in triangles, including proof of the Pythagorean Theorem, the sum of interior angles, base angles of isosceles triangles, midsegments, and medians, and apply these relationships to solve problems; and

(**E) prove a quadrilateral is a parallelogram, rectangle, square, or rhombus using opposite sides, opposite angles, or diagonals and apply these relationships to solve problems.**

Overview

In this lesson students examine three different formats for proofs: paragraph, flow-chart, and two-column proofs. Students reflect on the advantages of each format, and the connections between the different formats. They develop a list of criteria to judge the quality of a proof. Students construct a proof of a conditional statement about the diagonals of special quadrilaterals. Using the criteria developed in the lesson, they give and receive feedback on the proofs.

Instructional Strategies

- Expert and jigsaw groups
- Whole-class discussion

Lesson Objectives

- Students become familiar with paragraph, flow-chart, and two-column proofs
- Students develop the ability to critically judge proofs
- Students construct a proof about the diagonals of special quadrilaterals

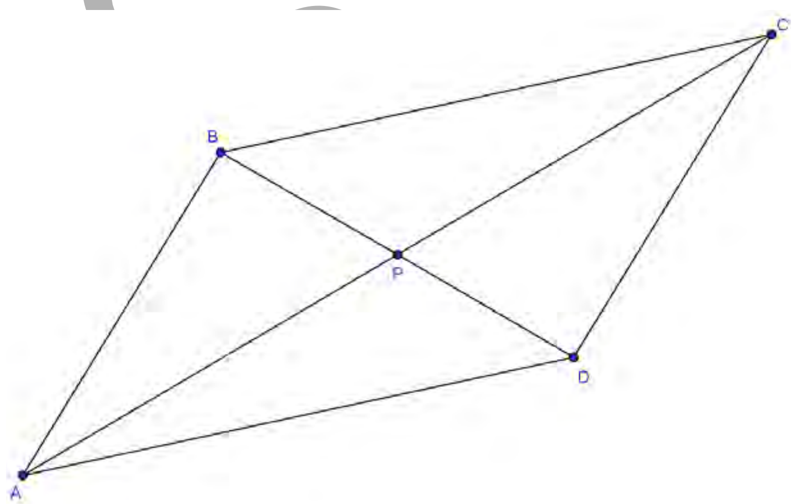
For **Teacher's Eyes Only**

There are three main formats for proofs:

- A **paragraph proof** presents the steps of the proof and their matching reasons as sentences in a paragraph.
- A **two-column proof** presents the steps of the proof in the left column and their matching reasons in the right column.
- A **flowchart proof** uses boxes and arrows to show the structure of a proof.

The following sections give examples of each proof format. The students will examine these proofs in the explore phase. Note that the reasoning differs slightly between the proofs in order to generate more discussion.

The following diagram illustrates the proofs. See the section Misconceptions for a discussion of drawing diagrams.



PARAGRAPH PROOF

Theorem: If a quadrilateral is a parallelogram, then its diagonals bisect each other.

Proof: Let ABCD be a parallelogram and P the intersection of the diagonals. We need to show that $AP \cong CP$ and $BP \cong DP$. Since $AB \parallel CD$, $\angle APD \cong \angle DCP$ because these are alternate interior angles. Similarly, $\angle ABP \cong \angle CDP$. Since $AB \cong CD$, $\Delta ABP \cong \Delta CDP$ by the AAS congruence property. Therefore, $AP \cong CP$ and $BP \cong DP$.

□

TWO-COLUMN PROOF

Given: *ABCD* is a parallelogram

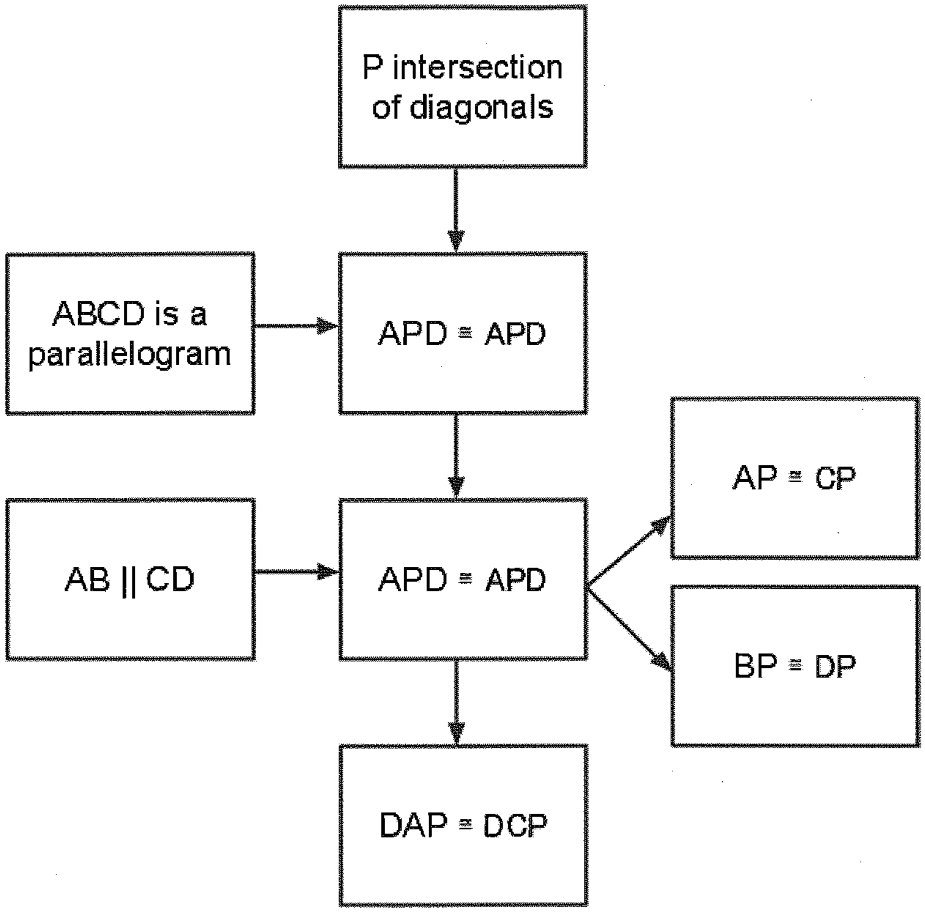
Prove: *AC* and *BD* bisect each other

Proof:

Label the intersection of the two diagonals with *P*.

Statement	Justification
$\angle APD \cong \angle DCP$	vertical angles
$\angle DAP \cong \angle BCP$	$AD \parallel BC$
$AB \cong CD$	*ABCD* is parallelogram
$\Delta ADP \cong \Delta CBP$	AAS
$AP \cong CP$ and $BP \cong DP$	$\Delta ADP \cong \Delta CBP$

FLOWCHART PROOF



COMPARING PROOF FORMATS

In itself, none of the formats is better or worse than the others. All formats can be used to write a good proof. However, there are important differences between the three different types.

The *paragraph proofs* reads the most naturally, and is the only format used in professional mathematics. However, constructing a paragraph proof requires practice and skill. There are few signposts to guide a beginning student.

The structure of a *two-column proof* provides more scaffolding than a paragraph proof does. Although two-column proofs are traditionally used in high-school geometry courses, its format is rigid and artificial.

Students may find the *flowchart proof* a good format to organize their initial thoughts when constructing a proof. The flexibility of its form makes it a good tool for the student constructing the proof, but makes it more difficult for another to read the proof later.

Summarizing, students benefit from writing a proof in three stages: flowchart to two-column to paragraph proof. Rather than the teacher stipulating a particular proof format, students should be

allowed to evaluate the advantages and disadvantages of each proof format for themselves.
GROUPING PROCEDURE OF THIS LESSON
This lesson uses a grouping procedure similar to the expert / jigsaw group teaching technique.

At the start of the explore phase, students are arranged in groups of three. These are the 'expert group'. Each expert group gets only one particular proof hand-out. The proof hand-out are distribute over the classroom in equal ratio. In the expert group the students study their given proof.

Once the students are comfortable with their particular proof, rearrange the students in new groups of three. These are the 'jigsaw groups'. Each jigsaw group consists of an expert on each of the different proof formats. In the jigsaw group the students first share their understanding of their respective proofs. Then they compare the proof formats, and come to a shared opinion on the advantages of each.

If the number of students in the class is one or two modulo three, you create one or two expert groups with four members. Some jigsaw groups will then have two experts on the same proof..

If you prefer your students to work in pairs, a grouping alternative is described in the section on explore phase.

Misconceptions

☒ Misconception

Students believe that the purpose of proof is to establish certain relationships in a given diagram (McCrone & Martin, 2009).

☑ Mathematics Concept

A proof establishes a theorem in general, not just for a particular case depicted by a given diagram.

🛠Rebuild Concept

Drawing a figure is an important problem-solving strategy (Polya, 2004). When the teacher or the textbook materials routinely provide diagrams, students do not develop the habit of creating their own drawings and illustrations. Therefore, the hand-outs of the proofs do not include a diagram on purpose.

REFERENCES
McCrone, S. S. & Martin, T. S. (2009). *Formal proof in high school geometry: Student perceptions of structure, validity, and purpose.* In D. A. Stylianou, M. L. Blanton, & E. J. Knuth

(Eds.), Teaching and learning proof across the grades: a K-16 perspective. New York, NY: Routledge, 204-219

Polya, G. (2004). *How to solve it: a new aspect of mathematical method.* Expanded edition. Princeton, NJ: Princeton University Press.

Student Prior Knowledge

This lesson builds on the Xtreem Geometry lesson: TEKS G.3 AC Conditional statements. Students need the understanding of conditional statements and their converses, inverses, and contrapositives gained in that lesson. Moreover, students need prior understanding of the following items:

- Triangle congruence properties
- Alternate interior angle theorem: if a transversal cuts two parallel lines, then alternate interior angles are congruent.
- Conversely, if the alternate interior angles formed by a transversal are congruent, then the two lines are parallel.

These prerequisites are covered in the Xtreem Geometry lesson: TEKS G.10 B, Larry's Building Company.

Materials

- Faulty proof pieces
- Hand-outs of proofs in different proof styles.

5 E's

OVERVIEW

The activities in the different phases, their duration, and groupings, are summarized in the following table:

Phase	**Duration**	**Activity**	**Grouping**
Engage	5	Faulty proof	Whole class
Explore	10 15	Examine proof Compare different proofs	Expert group Jigsaw group
Explain	10 10	Brain-storm criteria Discuss criteria	Jigsaw group Whole class
Elaborate	25	Construct draft of proof	Jigsaw group
Evaluate	15 15	Evaluate draft proofs Finalize proof	Jigsaw group Jigsaw group

ENGAGE

The learner is introduced to a new experience and must draw from prior experiences to make sense of the engage activity.

Using the faulty proof pieces (see the black line masters) and an overhead projector, present a 'proof' of the following (false) 'theorem' to the students:

> The square whose side is 21 cm has the same area as the rectangle whose sides are 34 cm and 13 cm.

The more students you can convince with flair and a flashy presentation, the better! Of course, this 'proof' is flawed. It is most likely that some students are not convinced, and that you can lead the class to realize this. If the students appear to accept the 'theorem', suggest that they verify the conclusion by computing the area of the square and of the rectangle. As soon as someone mentions that 21 x 21 = 441 but 34 x 13 = 422, conclude that you can only be sure of a proof if you have examined it closely and start the explore phase.

REFERENCE

Dubnov, Ya. S. (1965). *Mistakes in Geometric Proofs*. Boston, MA: D. C. Heath and Company.

EXPLORE

During the explore activity, the student becomes directly involved with a particular phenomena by manipulation of materials that are used to discover the phenomena.

In the explore phase, students examine a geometric proof in three different formats: as a paragraph proof, as a flow-chart proof, and as a two-column proof. All proofs establish the same theorem: *if* a quadrilateral is a parallelogram, *then* its diagonals bisect each other. This is one of the conjectures about the diagonals of special quadrilaterals that the students made in the previous lesson (Xtreem Geometry TEKS G.3 AC, Conditional Statements).

See the section Teacher's Eyes Only for the proofs and a comparison of the different formats, and an explanation of the grouping procedure used in this lesson.

1. Arrange the students in groups consisting of three students – these are the expert groups
2. Give each expert group one particular kind of proof to study: the paragraph proof, the two-column proof, or the flowchart proof (see black line masters)
3. Tell the expert groups to examine the proofs; suggest that they draw a diagram to illustrate the reasoning.
4. After about 10 minutes, rearrange the students in new groups of three students, consisting of one expert on each of the three proof formats – these are the jigsaw groups
5. The experts explain their respective proof to the other members of the group
6. After about 15 minutes, start the explain phase; the students remain in their jigsaw groups

GROUPING ALTERNATIVE

In the scenario described above, the students work in groups of three. If you prefer students to work in pairs, you can use the following alternative grouping scheme:

1. Arrange the students in pairs consisting of two students – these are the expert groups
2. Give each pair one particular kind of proof to study: either the paragraph proof or the two-column proof (see black line masters)
3. Tell the expert groups to examine the proofs; suggest that they draw a diagram to illustrate the reasoning.
4. After circa 10 minutes, rearrange the students in new pairs, consisting of one expert on the paragraph proof and one expert on the two-column proof – these are the jigsaw groups
5. The experts explain their respective proof to the other member of the group
6. After circa 10 minutes, hand out the flow-chart proofs
7. Each expert group studies the flow-chart proof together
8. After circa 5 minutes, start the explain phase; the students remain in their jigsaw groups

EXPLAIN

The student communicates in verbal and written form about the information derived from the learning experience.

During the explore phase, the students studied three different proof formats. The purpose of the explain phase is to reflect on the different formats. To start the explain phase, ask each group to discuss and write down the advantages and disadvantages of each type of format.

After a few minutes, have the students share their ideas with the whole-class by filling out a table on the blackboard or the overhead projector. The following table lists some examples:

Proof format	**Advantages**	**Disadvantages**
Paragraph proof	Reads naturally	Hard to connect statement and justification
Two-column proof	Statements and justifications clearly linked	Rigid structure
Flowchart proof	Flexible, helps you in organizing your thoughts.	Hard to read for someone else

Ask the students to write down several characteristics of a good proof. You may want to use the phrase 'a good proof is …' as a stem. Examples of possible endings are: correct, convincing, logical, detailed, precise, clear. Lead a classroom discussion, trying to achieve consensus on a list. Write down the list so that it can be referred to later.

The following list was reported by Stylianides & Stylianides (2009, unpaginated electronic publication):

1. The proof is correct.
2. The proof addresses the specific question or problem that was posed. It is focused, detailed, and precise. There are no irrelevant or distracting points.
3. The proof is clear, convincing, and logical. A clear and convincing proof is characterized by the following: (a) The proof uses language, representations, definitions that are understood by the people to whom the proof is addressed; (b) The proof could be used to convince a skeptic (not just oneself or a friend); (c) The proof does not require the reader to make a leap of faith (e.g., "This is how it is" or "You need to believe me"); (d) Key points are emphasized; (e) If applicable, supporting pictures, diagrams, and equations are used appropriately and as needed; (f) The proof is coherent; (g) Clear, complete sentences are used; and (h) The proof could be used by someone to solve a similar problem.

REFERENCE

Stylianides, A. J. & Stylianides, G. J. (2009). *Proof constructions and evaluations*. Educational Studies in Mathematics. Published online: March 6, 2009.

ELABORATE

During the elaboration phase, students expand their knowledge by making connections about what they have learned and applying this new knowledge to real world situations.

In the elaborate phase, each group of students constructs a proof of the converse of the conditional statement considered in the hand-outs.

After reaching a kind of consensus on the characteristics of a good proof, direct the students' attention back to the original conditional statement. The proofs that the students studied, established this statement as a theorem, but what about its converse: *if* the diagonals of a quadrilateral bisect each other, *then* the quadrilateral is a parallelogram? Remind the students that they could not find a counterexample to this statement in the previous lesson. Ask the students to find a proof of this statement.

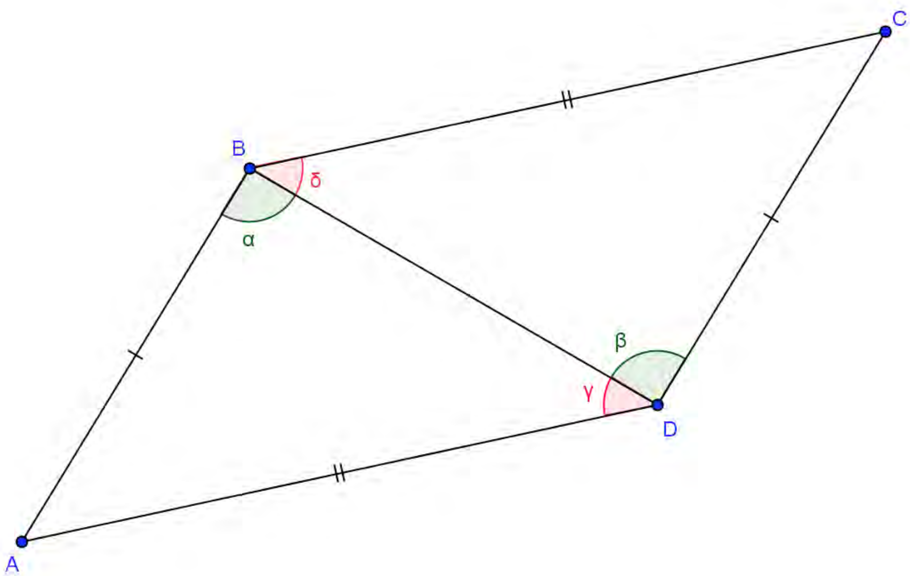
You should let the students choose the proof format they like. The hand-outs serve as examples of the different proof styles. Given below are examples of possible proofs. Note that these proofs use the following lemma:

> If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

A paragraph proof of this lemma is given below. You should lead to students to at least recognize that they use this lemma, and preferably have them prove it. If necessary, you can prove this lemma as a collaborative class effort.

Lemma: If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

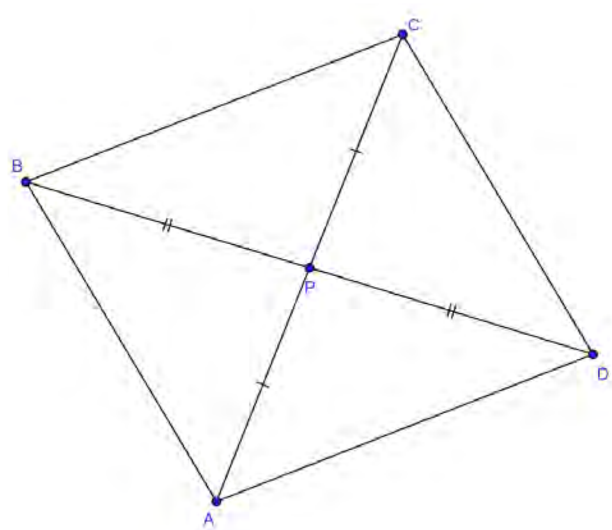
Proof: Let *ABCD* be a quadrilateral with $AB \cong CD$ and $BC \cong AD.$ By definition, a parallelogram is a quadrilateral with two pairs of parallel sides. We need to show that $AB \parallel CD$ and $BC \parallel AD.$ Draw the line segment *BD* and label the angles as in the following diagram:



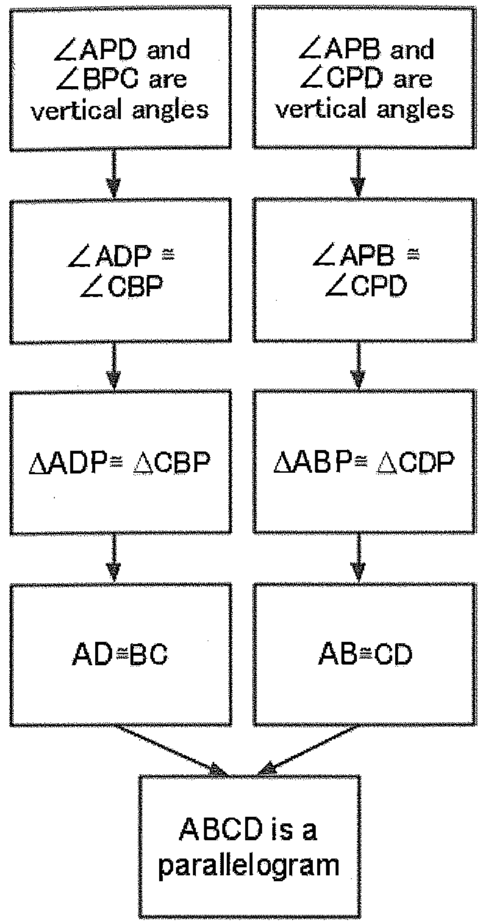
By the SSS congruence property, $\Delta DAB \cong \Delta BCD$. Hence, $\angle\alpha \cong \angle\beta$ and $\angle\gamma \cong \angle\delta$. Thus, $AB \parallel CD$ and $BC \parallel AD,$ and quadrilateral *ABCD* is a parallelogram.

□

The following three subsections present a flowchart, a two-column, and a paragraph proof of the statement: if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.



FLOWCHART PROOF



TWO-COLUM PROOF

Given: A quadrilateral in which the diagonals bisect each other

Prove: The quadrilateral is a parallelogram.

Proof: Label the vertices of the quadrilateral counterclockwise *ABCD*. Label the intersection of the two diagonals *P*.

Statement	Justification
$\angle APD \cong \angle BPC$	vertical angles
$\Delta ADP \cong \Delta CBP$	AAS
$AD \cong BC$	CPCTC
$\angle APB \cong \angle CPD$	vertical angles
$\Delta ABP \cong \Delta CDP$	AAS
$AB \cong CD$	CPCTC
ABCD is a parallelogram	lemma

PARAGRAPH PROOF

Theorem: If the diagonals of a quadrilateral bisect each other, *then* the quadrilateral is a parallelogram.

Proof: Label the vertices of the quadrilateral counterclockwise *ABCD*. Label the intersection of the two diagonals P. Since $\angle APD$ and $\angle BPC$ form a pair of vertical angles, $\angle APD \cong \angle BPC.$ Hence, $\Delta ADP \cong \Delta CBP$ and $AD \cong BC.$ Similarly, $AB \cong CD$. Therefore, *ABCD* is a parallelogram by the Lemma.

□

EVALUATE

Evaluation throughout the learning experience is an ongoing process and has a diagnostic function.

The purpose of the evaluate phase is to provide feedback on the proofs that the students have created, and improve the proof write-ups based on this feedback.

The students need to switch the drafts of their proofs with another group. Ask the students to provide constructive critique using the criteria developed. For example, students can rate the proof on whether it possess the characteristics of a good proof as developed in the explain phase. You should also give feedback.

Finally, each group writes a final version based on the input received. You may want to grade the final version. The table below provides an example rubric.

EXAMPLE RUBRIC

	Minimal	**Basic**	**Proficient**
Reasoning	Not all statements have an appropriate justification	All statements are justified, but the justification is not always completely clear	All statements are linked to clear justification
Communication	Proof is hard to read	Proof is readable	Proof reads easily
Representation	Notation is badly chosen, undefined; diagram is lacking	Notation is appropriate; diagram serves to	Notation is well-chosen, diagram clearly illustrates the proof
Convincing	The proof fails to convince even a friend	The proof would convince a friend, but not a critic	The proof would convince even a critic

Black-Line Masters

Paragraph Proof

Theorem: If a quadrilateral is a parallelogram, then its diagonals bisect each other.

Proof: Let ABCD be a parallelogram and P the intersection of the diagonals. We need to show that $AP \cong CP$ and $BP \cong DP$. Since $AB \parallel CD$, $\angle APD \cong \angle DCP$ because these are alternate interior angles. Similarly, $\angle ABP \cong \angle CDP$. Since $AB \cong CD$, $\Delta ABP \cong \Delta CDP$ by AAS congruence property. Therefore, $AP \cong CP$ and $BP \cong DP$.

Two-column Proof

Given: *ABCD* is a parallelogram

Prove: *AC* and *BD* bisect each other

Proof:

Label the intersection of the two diagonals with *P*.

Statement	Justification
$\angle APD \cong \angle DCP$	vertical angles
$\angle DAP \cong \angle BCP$	$AD \parallel BC$
$AB \cong CD$	*ABCD* is parallelogram
$\Delta ADP \cong \Delta CBP$	AAS
$AP \cong CP$ and $BP \cong DP$	$\Delta ADP \cong \Delta CBP$

Flowchart Proof